B or H as the basic field in electromagnetism

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In the past, **H** was considered as a basic field in electromagnetism and **B** as a derived quantity. In modern times the inverse situation prevails. We discuss the advantages and disadvantages of these possibilities and we develop for Maxwell's equations a manifestly covariant formalism using **E** and **H** as primary fields and **D** and **B** as derived quantities. [S1063-651X(96)08310-9]

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I. B VERSUS H

In order to obtain a determinate system of equations for the fields appearing in Maxwell's equations where **E**,**D**,**B**,**H** are the usual components of the electromagnetic field, ρ the density of charge, and **J** the vector current,

$$c\boldsymbol{\nabla}\times\mathbf{E}+\partial_t\mathbf{B}=\mathbf{0},\quad \boldsymbol{\nabla}\cdot\mathbf{B}=0, \tag{1a}$$

$$c\boldsymbol{\nabla}\times\mathbf{H}-\partial_t\mathbf{D}=c\mathbf{J},\quad \boldsymbol{\nabla}\cdot\mathbf{D}=\rho, \tag{1b}$$

it is necessary to append certain constitutive relations, the form of which is dependent upon the nature of the material in which the electric and the magnetic fields occur. For instance, in a rigid, linear, stationary, nonconducting dielectric, constitutive relations as given by Maxwell [1] are

$$\mathbf{D}=\varepsilon\cdot\mathbf{E},\quad \mathbf{B}=\mu\cdot\mathbf{H}. \tag{2}$$

ε,μ are the permittivity and the permeability tensors and these relations imply that **E**,**H** are basic fields, **D**,**B**, derived quantities. The relations (2) are consistent with the fact [2] that the magnetic induction **B** plays a role in magnetic phenomena analogous to that of the displacement vector **D** in electrical phenomena and that the magnetic field strength vector **H** can be defined as the mechanical force which the magnetic field exerts on a fictitious magnetic monopole (a best definition of **H** is obtained in terms of Ampère's law). Nowadays people write

$$\mathbf{D}=\varepsilon\cdot\mathbf{E},\quad \mathbf{H}=\mu^{-1}\cdot\mathbf{B}, \tag{2a}$$

making of **B** a basic field.

As long as constitutive relations keep the previous simple forms, the difference between (2) and (2a) is semantic but not in more general media (think of ferromagnetism). So it is interesting to analyze the reasons leading one to prefer (2a).

First the manifestly covariant description of electromagnetism used in relativity [3] needs two antisymmetric tensors $F_{\mu\nu}(\mathbf{E},\mathbf{B})$ and $G_{\mu\nu}(\mathbf{D},\mathbf{H})$ ($\mu,\nu=0,1,2,3$), and Maxwell's equations read

$$\partial^\mu \tilde{F}_{\mu\nu}=0,\quad \partial^\mu G_{\mu\nu}=J_\nu. \tag{3}$$

$\tilde{F}_{\mu\nu}$ is the dual tensor $\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, where $\varepsilon_{\mu\nu\alpha\beta}$ is the permutation tensor and J_μ the four vector $(\mathbf{J},\rho)$. The general covariant linear constitutive relations are [4]

$$G^{\mu\nu}=\chi^{\mu\nu\alpha\beta}F_{\alpha\beta}, \tag{4}$$

where $\chi^{\mu\nu\alpha\beta}$ is a tensor with 21 independent components making clear that in this formalism **E** and **B** are the basic fields.

A second reason often put forward [3] is that Eqs. (1a) can be solved formally by expressing **E** and **B** in terms of potential functions while Eqs. (1b) cannot be solved until the derived fields **D**,**H** are known in terms of **E** and **B**. Let us now dispose of this argument on the simple example of an isotropic medium moving uniformly along the z axis with the velocity **u**.

We first assume that in the rest frame the constitutive relations have the form (2) (ε,μ are now scalar),

$$\mathbf{D}'=\varepsilon\mathbf{E}',\quad \mathbf{B}'=\mu\mathbf{H}'. \tag{5}$$

Using the transformations of electromagnetic fields under a Lorentz boost (see Appendix A) the relations (5) become in the laboratory frame [5]

$$\mathbf{D}=\varepsilon\Lambda\mathbf{E}+\mathbf{N}\mathbf{u}\times\mathbf{H},$$

$$\mathbf{B}=\mu\Lambda\mathbf{H}-\mathbf{N}\mathbf{u}\times\mathbf{E}. \tag{6}$$

We use the following notations. Any field A is decomposed into the direct sum of a 2D transverse component A_T and of a longitudinal component A_z along z: $A=A_T\oplus A_z$ and I denotes the $2\otimes 2$ identity matrix ($n^2=\varepsilon\mu$, $\beta^2=u^2c^{-2}$):

$$\Lambda=(1-\beta^2)(1-n^2\beta^2)^{-1}I\oplus 1,$$

$$N=(n^2-1)(1-\beta^2n^2)^{-1}I\oplus 0. \tag{7}$$

Substituting (6) into (1) gives the equations

$$c\boldsymbol{\nabla}\times\mathbf{E}+\mu\Lambda\partial_t\mathbf{H}-\mathbf{N}\partial_t\mathbf{u}\wedge\mathbf{E}=\mathbf{0},$$

$$\mu\boldsymbol{\nabla}\cdot\Lambda\mathbf{H}-\boldsymbol{\nabla}\cdot\mathbf{N}\mathbf{u}\wedge\mathbf{E}=\mathbf{0},$$

$$c\boldsymbol{\nabla}\times\mathbf{H}-\varepsilon\Lambda\partial_t\mathbf{H}-\mathbf{N}\partial_t\mathbf{u}\wedge\mathbf{H}=\mathbf{J},$$

$$\varepsilon\boldsymbol{\nabla}\cdot\Lambda\mathbf{E}+\boldsymbol{\nabla}\cdot\mathbf{N}\mathbf{u}\wedge\mathbf{H}=\rho, \tag{8}$$

and recently Tai [6] using the pseudotime variable $\tau=t+\beta c^{-1}\nu z$, where ν is the coefficient of I and the definition (7) of N was able to put these equations in a form that can be solved by conventional methods providing a simple and elegant solution of Eqs. (8).

Let us now take (2a) as constitutive relations in the rest frame

$$\mathbf{D}'=\varepsilon\mathbf{E}',\quad \mathbf{H}'=\kappa\mathbf{B}',\quad \kappa=\mu^{-1}. \tag{9}$$

These relations become in the laboratory frame

$$\mathbf{D}=\varepsilon K_d\mathbf{E}+\mathbf{K}(\mathbf{u}\times\mathbf{B}),$$

$$\mathbf{H}=\kappa K_h\mathbf{B}+\mathbf{K}(\mathbf{u}\times\mathbf{E}), \tag{10}$$

$$\mathbf{K}_d=(1-\beta^2n^{-2})(1-\beta^2)^{-1}I\oplus 1,$$

$$\mathbf{K}_h=(1-\beta^2n^2)(1-\beta^2)^{-1}I\oplus 1,$$

$$\mathbf{K}=(1-\beta^2)^{-1}I\oplus 0. \tag{10a}$$

Maxwell's equations (1a) are left unchanged and can be solved in terms of potential functions while substituting (10) into (1b) gives ($\mathbf{1}_z$ is a unit vector in the z direction)

$$c\boldsymbol{\nabla}\times K_h\mathbf{B}-\mathbf{K}_d\partial_t\mathbf{E}-\alpha(c\partial_2\mathbf{E}-c\mathbf{1}_z\boldsymbol{\nabla}\cdot\mathbf{E}+\mathbf{1}_z\partial_t\mathbf{B})=\mathbf{J},$$

$$\boldsymbol{\nabla}\cdot\mathbf{K}_d\mathbf{E}-\alpha(\boldsymbol{\nabla}\times B)_z=\rho,\quad \alpha=(\varepsilon-\kappa)(1-\beta^2)^{-1}\beta, \tag{11}$$

but there is no simple solutions of Eqs. (11) because they depend on three operators $\mathbf{K}_d$,$\mathbf{K}_h$,$\mathbf{K}$ instead of two $\boldsymbol{\Lambda}$,$\mathbf{N}$ for Eqs. (8).

This result shows that in this problem **H** should be considered as the basic component and dismisses the second argument in favor of **B**. There is no guarantee that the formal solution of Eqs. (1a) in terms of potential functions would make easier the general solution of Maxwell's equations.

Then, to dispose of the first argument in favor of **B** as the basic field we have still to prove that there exists a manifestly covariant formalism of electromagnetism using **E** and **H** as basic fields.

II. MANIFESTLY COVARIANT DESCRIPTION OF MAXWELL'S EQUATIONS

We introduce two antisymmetric tensors $F'_{\mu\nu}(\mathbf{E},\mathbf{B}),G'_{\mu\nu}(\mathbf{D},\mathbf{B})$ obtained by exchanging **H** and **B** in the usual tensors $F_{\mu\nu}$ and $G_{\mu\nu}$. Then, we consider the self-dual tensors

$$K_{\mu\nu}=F'_{\mu\nu}+(i/2)\varepsilon_{\mu\nu\alpha\beta}F'^{\alpha\beta},$$

$$L_{\mu\nu}=G'_{\mu\nu}+(i/2)\varepsilon_{\mu\nu\alpha\beta}G'^{\alpha\beta}. \tag{12}$$

whose components are, respectively, the complex vectors $\mathbf{P}=\mathbf{E}+i\mathbf{H}$, $\mathbf{Q}=\mathbf{D}+i\mathbf{B}$ $(i=\sqrt{-1})$. More precisely,

$$P_j=K_{0j}=-K_{j0}=(i/2)\varepsilon_{jkl}K^{kl},$$

$$Q_j=L_{0j}=L_{j0}=(i/2)\varepsilon_{jkl}L^{kl}. \tag{13}$$

In these relations, which follow from the isomorphism between the Lorentz group and the 3D complex rotation group [7], the latin indices take the values 1, 2, 3, and ε_{ijk} is the permutation tensor. Note that the real part of the vectors **P**,**Q** is a polar vector while the imaginary part is an axial vector.

The summation of Eqs. (1a) with Eqs. (1b) multiplied by i gives

$$ic\boldsymbol{\nabla}\times\mathbf{P}+\partial_t\mathbf{Q}=-c\mathbf{J},\quad \boldsymbol{\nabla}\cdot\mathbf{Q}=\rho \tag{14}$$

and to write (14) in a manifestly covariant form we introduce the antisymmetric tensor $M_{\mu\nu}$ with the components

$$M_{0\mu}=L_{0\mu},\quad M_{\mu 0}=L_{\mu 0},\quad M_{ij}=K_{ij}, \tag{15}$$

that is,

$$M_{\mu\nu}=\begin{vmatrix} 0 & -Q_x & -Q_y & Q_z \\ Q_x & 0 & iP_z & -iP_y \\ Q_y & -iP_z & 0 & iP_x \\ Q_z & iP_y & -iP_x & 0 \end{vmatrix} \tag{15a}$$

and the two four vectors

$$\partial_\mu=(\partial_j,\partial_t),\quad J_\mu=(J_j,\rho). \tag{16}$$

It is easy to check that Eqs. (14) take the simple form discussed in Appendix B,

$$\partial^\nu M_{\mu\nu}=-J_\mu, \tag{17}$$

which is an alternative manifestly covariant description of electromagnetism. The general linear covariant constitutive relations can be imposed in a similar way to (4),

$$G'^{\mu\nu}=\xi^{\mu\nu\alpha\beta}F'_{\alpha\beta} \tag{18}$$

and the tensor ξ has the same symmetry properties as the tensor κ.

III. CONCLUSION

A definitive answer to the question of considering **B** or **H** as a basic component is difficult to assess and depends probably on the physical process under discussion. For instance, in ferromagnetism where the relation between **B** and **H** is a nonlinear functional $\mathbf{B}=f(\mathbf{H})$ (hysteresis loop), **H** is the natural basic quantity. Similarly the example discussed in Sec. I suggests the same conclusion for chiral media. At the opposite **B** plays the role of basic field in a process requiring the Lorentz force but one must take **H** as fundamental for the Ampère force. In some cases there may exist some doubts; for instance [2], the question of whether **H** or **B** is the vector to be used for the mechanical force exerted by a magnetic field on magnetized matter has been only partly answered [8].

Generally calculations are easier when mathematics and physics share the same symmetries. A good example is provided by the Tai result. So it is important to use in the constitutive relations the physically fundamental fields. In addition, when **H** is the basic field, the 3D complex formalism used in Sec. II performs very well. We were able, for instance, to simplify somewhat Tai's expressions and to extend his result to bi-isotropic Tellegen media.

APPENDIX A

The components of the electromagnetic field in two frames moving apart with the relative uniform velocity $\mathbf{u}$ along $0z$ satisfy the relations [9]

$$\mathbf{E}'=K_0(\mathbf{E}+\mathbf{u}\times\mathbf{B}),\quad \mathbf{B}'=K_0(\mathbf{B}-\mathbf{u}\times\mathbf{E}),$$

$$\mathbf{D}'=K_0(D+\mathbf{u}\times\mathbf{H}),\quad \mathbf{H}'=K_0(\mathbf{H}-\mathbf{u}\times\mathbf{D}),\tag{A1}$$

$$K_0=(1-\beta^2)^{-1/2}I\oplus 1.\tag{A2}$$

APPENDIX B

One checks easily that Eq. (17) supplies the continuity condition $\partial^\mu J_\mu=0$. Now let $Z_\mu=X_\mu+iY_\mu$ be a complex potential and $M_{\mu\nu}=\partial_\mu Z_\nu-\partial_\nu Z_\mu$. Imposing the generalized Lorentz condition $\partial^\mu Z_\mu=0$, we get from (17) the equations

$$\partial^\nu\partial_\nu X_\mu=J_\mu,\tag{B1a}$$

$$\partial^\nu\partial_\nu Y_\mu=0.\tag{B1b}$$

Using the Green's function allows us to solve formally the inhomogeneous wave equation (B1a). For instance, in free space

$$X_\mu(\mathbf{x},t)=\int\delta(t-t'-|\mathbf{x}-\mathbf{x}'|/c)|\mathbf{x}-\mathbf{x}'|^{-1}\times J_\mu(\mathbf{x}',t')d^3x'dt',\tag{B2}$$

where δ is the Dirac distribution. The solution of the homogeneous wave equation (B1b) is obtained by the usual methods depending on the problem under discussion.

From the definition of $M_{\mu\nu}$ and Z_μ we get

$$B_j=-\partial_0 Y_j-\partial_j Y_0\tag{B3}$$

so that using the Lorentz condition $\partial^\mu Y_\mu=0$ and the equation (B1b) one checks easily that the divergence equation (1a) is satisfied. Note that in this covariant formalism the Lorentz condition is a necessary condition.

Finally Eq. (17) may be obtained from the covariant Lagrangian,

$$L=\{1/32\pi M_{\mu\nu}M^{\mu\nu}-1/2cJ^\mu Z_\mu\}+\{\text{c.c.}\},\tag{B4}$$

where {c.c.} denotes the complex conjugate of the quantities inside the preceding bracket. This covariant formalism of electromagnetism could be of interest in quantum field theory.

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